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Prospective teachers work on defining quadrilaterals through an exploratory approach

Abstract: This research addresses elementary school prospective teachers' education in geometry and takes place in Portugal, in the context of a geometry course taught by the first author of this paper. We aim to characterize the activity of prospective teachers in defining quadrilaterals as they work on exploratory tasks in a design-based research study for a geometry course at the 2nd year of their teacher education program. The teaching-learning conjecture is based on the relevance of exploratory work, also valuing prospective teachers' reflection on their learning. We also assume that the activity of defining quadrilaterals plays an important role in developing reasoning, deepens the knowledge of its properties and supports a more complete view of mathematics. Data was gathered from the participants' reports and portfolios, as well as audio and video records of classroom discussions. The results show that initially the participants had misconceptions about what a definition is, just focusing on the necessary properties of the figures. However, at the end of the sequence of tasks, they were able to take into account that the set of properties has to be sufficient to identify the defined figure, revealing the development of their knowledge. At the end of the sequence, most participants presented correct definitions, using properties that they previously ignored and showing comprehension of the underlying concepts. They produced economical definitions in few cases, and performed better in inductive than deductive reasoning. We conclude that the exploratory work allowed participants to construct their knowledge in a meaningful way and reflection played an important role in making them aware of personal preconceptions and knowledge. In the beginning, the participants

Key words: teacher education, geometry, geometric reasoning, quadrilaterals, definitions, exploratory approach.

analysed definitions strictly focusing on the necessary properties of the figure, which seemed to be simple when the properties related to visible elements or were well known facts. In the final stage of the sequence of tasks, the participants also took into account that the set of properties has to be sufficient to identify the defined figure and, in the construction of definitions, they presented mostly correct definitions using properties that they previously ignored, showing an understanding of the underlying concepts and properties. Doing this process after investigating and classifying quadrilaterals encouraged the prospective teachers to mobilize different kinds of properties and to reason according to the established classifications, which was a very challenging activity.

1 Introduction

In Portugal, the education programs of prospective teachers from kindergarten to elementary school lasts for approximately five years. These programs include preparation in educational sciences and scientific and didactic preparation in mathematics, language, sciences, arts and sports, dedicating just a small amount of time to each area. Most of these programs are run in schools of education, which staff is simultaneously responsible for the scientific and didactical preparation of prospective teachers. The candidates enter these programs with different backgrounds, namely in mathematics, as most studied math until the 9^{th} grade, and only few until the 12^{th} grade. Recent studies in our country show less than satisfactory results concerning the geometric knowledge prospective elementary teachers' present before but also after attending their teacher education programs (Menezes, Serrazina & Fonseca, 2014; Tempera, 2010). Particularly concerning the knowledge of geometric figures, the study of Tempera (2010) claims that prospective teachers are much attached to the prototypes they acquired earlier and that position, aspect and size seem to overlap the properties of a class of figures. A similar conclusion is also found in studies from other countries, concerning teachers and prospective teachers, indicating that geometry is an area in which they perform poorly, show weak geometric vocabulary and have little self-confidence (Clements & Sarama, 2011; Fujita & Jones, 2006; Jones, Mooney & Harries, 2002).

The preparation of future teachers also suffers from the fact that mathematics educators have very different views about what geometry can or should be taught in teacher preparation courses (Jones, 2000). This is problematic as the success of the teachers' work depends, to a great extent, on their deep understanding of geometry. In addition, we must take into account that knowing geometry does not ensure effectiveness, how teachers come to know it matters as well (Jones, Mooney & Harries, 2002).

This situation challenges us to seek ways of improving the teacher's education in this area, specifically in a curricular unit of geometry taught in the school of education where the first author of the paper teaches. This unit is the only one exclusively dedicated to geometry and takes place in the 2^{nd} year of the teacher education program, corresponding to 50 hours of work in class (distributed in two classes per week for three and a half months). To improve prospective teachers' geometry learning, we developed a design research experiment in the context of a curricular unit based on exploratory work, linking geometry and didactics and valuing prospective teachers' reflection on their learning. In this context, the goal of this paper is restricted to characterize the activity of prospective elementary school teachers in defining, a key process of geometric reasoning.

2 Conceptual framework

2.1 Prospective elementary teacher education in geometry

For the National Council of Teachers of Mathematics [NCTM] the knowledge necessary for teaching includes "the content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality" (NCTM, 1991, p. 132).

This perspective is coherent with the idea advocated by Ma (1999) that teachers need a profound understanding of fundamental mathematics. But what does this mean in geometry? The NCTM (1991) states that all teachers should understand how geometry is used to describe the world we live in and how it is used to solve concrete problems; analyse a diverse set of two and three dimensional figures; use synthetic geometry, coordinates and transformations; improve their skills in producing arguments, justifications and privilege spatial visualization. The Conference Board for the Mathematical Sciences [CBMS] (2000) proposed that prospective K-5 teachers must develop competence in the following areas: Visualization skills (projections, cross-sections, decompositions; representing 3D objects in 2D and constructing 3D objects from 2D representations); basic shapes, their properties, and relationships among them (angles, transformations, congruence and similarity); and communicating geometric ideas (learning technical vocabulary and understanding the role of mathematical definition). The most recent report of CBMS (2012) updates the main ideas for teaching preparation in geometry, presenting less topics and less complex competencies:

- Understanding geometric concepts of angle, parallel, and perpendicular, and using them in describing and defining shapes; describing and reasoning about spatial locations (including the coordinate plane).
- Classifying shapes into categories and reasoning to explain relationships among the categories.
- Reason about proportional relationships in scaling shapes up and down. (p. 30)

This shift illustrates the lack of agreement about the geometric knowledge that teachers must hold.

In addition, the education of teachers concerns also the ways they are taught. Regarding the results of several studies about prospective teachers' knowledge of mathematics, Watson and Mason (2007) propose that courses should prompt prospective teachers to engage in mathematical thinking through working on suitable mathematical tasks, develop their understanding about the features and power of those tasks, reflect on the experience of doing mathematics tasks individually or with others, challenge approaches dominated by procedures which depend on rote memorization, and observe and listen to learners. These orientations are also consistent with ideas underlined by other researchers, according to whom prospective teachers should learn using the same methods that are recommended they should use in the future (Ponte & Chapman, 2008); connecting subject matter knowledge and pedagogy is a promising strategy to develop both kinds of knowledge and their integration, which is critical to teach well (Ball, 2000).

The present work follows these proposals, as we focus on prospective teachers' learning as they work on exploratory tasks and reflect on their own learning. Exploratory tasks demand learners to engage actively in the construction of their knowledge by solving tasks where there is no clear solving method, as it is the case of problems. Sometimes, learners are also challenged to ask questions or extend the purpose of the task, as in the case of investigations. They need to interpret the given information, develop strategies, represent and communicate their solutions. This promotes their understanding of representations, concepts, and procedures, and also develops the ability to argue about ideas, as they communicate such ideas to others. Work on exploratory tasks develops usually in three phases (Ponte, 2005): (i) presenting and interpreting the task; (ii) carrying out the task individually, in pairs, or in small groups; and (iii) presenting and discussing results and making a final synthesis.

2.2 Geometric reasoning and the process of defining

The study of geometry is the natural context to develop and use visualization, spatial reasoning and geometric modelling to solve problems (NCTM, 2000). Despite the growing focus on geometric reasoning and visualization in research, the clarification of the meanings of these notions is still lacking (Gutiérrez, 1996). This is even more complicated by the many expressions often used with very similar meanings (e.g., geometric reasoning, visual reasoning, visualization, spatial thinking…). For example, for Battista (2007) "geometric reasoning consists, first and foremost, of the invention and use of formal conceptual systems to investigate shape and space" (p. 843), a definition we may find too broad. Also, the van Hiele model cover different forms of reasoning as it seeks to describe how individuals' geometric reasoning develops through five levels: 1) visual-holistic reasoning; 2) descriptive-analytic reasoning; 3) relational-inferential reasoning; 4) formal deductive proof; and 5) rigor (Battista, 2009). So our interest in investigating the development of geometric reasoning drove us to ask what is specific of this kind of reasoning and what are its what main features. A possible approach to study geometric reasoning consists in analysing it from its processes, which are present in other areas but have some specificity in geometry. One of these processes is defining.

Before we discuss the relationship between the process of defining and geometric reasoning, let us focus on the decisive role of definitions in mathematics. As Veloso (1998) points out, "without definitions there is no mathematics and mathematical communication is not possible" (p. 375). Looking to systematize the roles of definitions presented in literature, Zaslavsky and Shir (2005) feature four aspects: (1) introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties; (2) constituting fundamental components for concept formation; (3) establishing the foundation for proofs and problem solving; and (4) creating uniformity in the meaning of concepts, which allows us to communicate mathematical ideas more easily. From the perspective of the logical principles definitions should meet, Winicki-Landman and Leikin (2000) present five main aspects referred by many mathematicians:

(1) Defining is giving a name. The name of the new concept is presented in the statement used as a definition and appears only once in this statement.

(2) For defining the new concept, only previously defined concepts may be used.

(3) A definition establishes necessary and sufficient conditions for the concept.

(4) The set of conditions should be minimal.

(5) A definition is arbitrary. (p. 17)

In the context of geometry, de Villiers, Govender and Patterson (2009) add that definitions may be inclusive or exclusive, which is a consequence of the mutual relationship between the processes of defining and classifying, as "the classifications of any set of concepts implicitly or explicitly involves defining the concepts involved, whereas defining concepts in a certain way automatically involves their classification" (p. 191). So, a definition is considered inclusive "if allows the inclusion of more particular concepts as subsets of the more general concept" (p. 191); it is exclusive if "the concepts involved are considered disjoint from each other" (p. 191). Historically, inclusive definitions were not always preferred over exclusive, as it is the case of the definitions of quadrilaterals proposed by Euclid, but for de Villiers et al. (2009), this type of definitions has several advantages, so they suggest its preference. From the mathematical point of view, the use of inclusive definitions promotes some economy in the construction of other definitions – for example, for the parallelepiped definition it is sufficient to say that the faces are parallelograms, thus including rectangles, rhombus and squares – and an economy in the production of theorems – if a property is demonstrated for a class of objects, it is automatically established for all objects that are part of this class. Also in problem solving, inclusive definitions may enlarge the set of solutions.

In fact, as stated by Zaslavsky and Shir (2005), there are some aspects that are "imperative"to consider in the construction of definitions, such as being unambiguous or not containing contradictory statements, but there are others non-consensual, so they may be considered "optional". The imposition of the set of conditions to be minimal, that is, the definition is "economic" such as suggested by de Villiers et al. (2009), it is a controversial aspect, which is probably due to differences of opinion about what constitutes a good definition. Thus, we can say there is no "definition" that is commonly accepted by mathematicians on what constitutes a good mathematical definition.

So why should we consider defining as a process of geometric reasoning? According to Mariotti and Fischbein (1997), "defining in the geometrical field seems to present a great complexity due both to general characteristics of the defining process and specific characteristics of the geometrical concepts" (p. 224). When geometrical concepts are concerned, defining requires a simultaneous double movement between the figural and the conceptual level, going through several main steps that may be summarized as follows:

> Observing; identifying the main characteristics; stating properties according to these characteristics; returning to observation, chec-

> king the definition with regard to figural differences, and so on . . . The process of elaborating a definition consists of a double process from the particular to the general and vice versa, from the general to the particular. (pp. 226-227)

Thus, the construction of a definition for a set of objects implies identifying the attributes that are common to all objects and make a generalization to formulate a necessary condition, using inductive reasoning; for the condition to be sufficient, it must lead to the set of objects intended, which involves deductive reasoning. So, it seems clear why Mariotti and Fischbein (1997) argue that the process of defining must be considered both as a component of geometrical reasoning and a specific process of mathematical activity, holding a constructive and creative role. In addition, the construction of economical definitions proposed by de Villiers et al. (2009) mobilizes deductive reasoning to a greater extent, as for the definition to be economic, it is necessary to ensure that none of the conditions presented can be deduced from another. This reinforces the claim from those researchers that defining is so import as solving problems, conjecturing or proving and that, despite its relevance, is much neglected in mathematics teaching.

In fact, in the school context, definitions are often presented to pupils when their knowledge has not evolved enough to integrate them in a natural way. This tendency is accentuated by the conflict between how mathematical knowledge is organized and how it is created and evolves (de Villiers et al., 2009), as well as how it is learned (Vinner, 1991). In mathematics, a theory starts with primitive terms and axioms, from which all other concepts are defined and theorems are proved, using the rules of logical reasoning. However, this path does not reflect the way mathematics is created and how it is learned, although there is much evidence that this view tends to contaminate the teaching of mathematics. As Veloso (1998) points, concepts must be built slowly and gradually, unlike the artificial imposition of definitions that do not have the absolute character often attributed to it. Pupils should familiarize themselves gradually with the definitions, which should result of their mathematical experience rather than precede it. The opposite orientation leads pupils to memorize the definitions, but inevitably they end up revealing difficulties in reason according to them (de Villiers et al., 2009; Mariotti & Fischbein, 1997). Moreover, the fact that pupils get used to dealing with definitions without ever discussing the concept of definition or even look at examples of alternative definitions, promotes the construction of misconceptions about the meaning and role of definitions (Zaslavsky & Shir, 2005).

Some studies involving pupils from different ages and academic backgrounds and prospective teachers show the importance of analysing and producing definitions. For instance, in a study with grade 12 pupils, Zaslavsky and Shir (2005) show that they may struggle with the arbitrary character of the definitions, as they did not accept a set of necessary and sufficient properties for a definition when it referred invisible elements, such as diagonals. The activity of analysing possible definitions led them to discuss the characteristics and the role of definitions, and showed itself to be a powerful tool in the development of reasoning and understanding of the underlying concepts. Besides, that activity promoted a more humanistic view of this science by letting aspects such as personal preferences to play a role in deciding which definitions are considered better. In another study involving prospective secondary teachers, Zazkis and Leikin (2008) report that prospective teachers' ability to generate examples of definitions varies considerably, even when the task relates to a familiar concept such as a square. Also, the construction and analysis of definitions for square showed the participants' ability to distinguish necessary and sufficient conditions, use adequate language, and show conceptions about defining. Finally, these researchers emphasize that, for teachers to be able to support pupils in this process, they need to be competent in performing it, an idea that is in the centre of this study.

3 Methodology

This study is based on an intervention aiming to change practices and enhance prospective teachers' preparation in geometry. The participants are 90 infant education and elementary school prospective teachers attending a geometry course in their 2nd year of studies. Serious weaknesses of geometric knowledge of these teacher candidates was diagnosed in previous studies (Tempera, 2010). With respect to geometry and particularly regarding quadrilaterals, the national curriculum in place when the prospective teachers where in elementary and high school suggested the study of some properties in parallelograms (specifically sides and axes of symmetry). However, it did not suggest that pupils should be involved in the process of defining and there was no mention about the hierarchical organization of quadrilaterals. In addition, most of the textbooks available in schools did not promote exploratory work focusing on those figures and their properties.

The research focus is on learning in context, starting from the conception of strategies and teaching tools, using the methodology of design-based research, in the form of a preservice teacher development experiment (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) in which the teacher also plays the role of the researcher. We expect to run through cycles of creation and revision, trying to deal with the problems that we will find along the way. At the present time, two cycles were conducted involving 90 prospective teachers (60 in the first cycle and 30 in the second) from three different classes, almost all of them female.

The design of the experiment is driven by a conjecture with different levels of specificity. At a more general level, we follow the perspective that the education of preservice teachers must be consistent with the methodologies we advocate for their future practices, so we adopted an exploratory teaching approach and underlined the value of reflection. At a more particular way, the conjecture addresses the specific mathematical topic – the study of quadrilaterals – and emphasizes the role of defining in the development of reasoning, in learning the properties of figures and also in developing a better understanding of the nature of mathematics. Finally, we conjecture that the activity of defining quadrilaterals should be performed after investigating its properties and classifying them.

Driven by this conjecture, the study of quadrilaterals was developed in six lessons, following three sequential steps: (i) to investigate the properties of quadrilaterals using the dynamic geometry environment (DGE) *Geogebra*; (ii) to classify quadrilaterals; and (iii) to define quadrilaterals. In the first step, the prospective teachers worked on an investigative task where they discovered the properties of special quadrilaterals (square, rectangle, rhombus, parallelogram, trapezium, and kite) using *Geogebra*, by manipulating draggable figures previously constructed by the teacher and registering all the properties, namely concerning sides, angles, diagonals, and symmetry. In the second step, they classified the figures using a flowchart and a Venn diagram with the purpose of realizing that different criteria lead to different organizations. In the third step, the participants worked on a definition task which is the focus of this paper. In all these lessons, the task was first introduced collectively by the teacher, then the participants worked in small groups[1] and registered their answers, which were finally discussed collectively at the end of the lesson. Sometimes, the participants were asked to work on the task individually for a short period of time, and then share the findings with the other members of the group.

Data gathered includes the written records of the solutions of the tasks from the 90 participants (namely two questions from the diagnostic test and

[1]In this institution the classes have 30 prospective teachers approximately and it is common to work in small groups (four or five elements). They choose their own group and usually they maintain that organization in other curricular units.

the definition task, both solved in classroom), audio records from interviews to four participants, audio and video records of group work and collective oral discussions, and participants' reflections collected from portfolios.

The analysis focuses on the work about definitions, regarding two different kinds of activities: analysing given definitions and constructing definitions. For the first activity, we carried out two types of analysis, depending on the nature of the data. For the quantitative data (relating to the question in diagnostic test), we considered the correction of the responses, the type of properties associated with these responses (namely, its relation to visible or hidden elements) and the concept of definition they conveyed. For the data collected from interviews and dialogues between the participants, the categories emerged from the data in relation to the focus of analysis of the prospective teachers (Table 1). If the focus of the analysis is on the necessary character of the properties stated by the definition, we have to verify those properties. Sometimes, they can be immediately accepted because they are facts already acquired (e.g. all sides are equal in a square); other times, it is necessary to resort to the representations of the figures (physical or mental). To verify that a set of properties is sufficient, it is necessary to identify the figures generated by those properties and check if it corresponds to the class of figures to be defined. Finally, if the above conditions are checked, we may focus on the economic aspect and verify that the set of properties is minimal. In this case, the response may result from two kinds of reasoning: (i) we try to deduce one of the properties from the other(s) or (ii) we generate new figures from a reduced set of properties and examine if the class remains the same.

Focus of analysis	**Category**
The properties are necessary	Verify properties
The properties are sufficient	Generate figures
The set of properties is minimal	Deduce one of the properties from the other(s) Generate new figures from a reduced set of properties

Table 1. Categories for the activity Analysing definitions.

Regarding the process of constructing definitions, we adopted the categorization of de Villiers' et al. (2009): economical definitions, correct definitions and incorrect definitions (Table 2). In this last case we considered definitions containing necessary properties but insufficient to define the intended quadrilateral; this category also includes the definitions presenting properties that do not apply to some or all objects. Correct definitions present properties necessary and sufficient; if the set of those properties is minimal, the definition is economical.

Features of the constructed definition	**Category**
The definition present properties that do not apply to some or all elements of the class; Definitions presenting necessary but insufficient properties.	Incorrect definition
Definitions presenting necessary and sufficient properties.	Correct definition
Definitions presenting a minimal set (or "barely-not-minimal") of necessary and sufficient properties.	Economic definition

Table 2. Categories for the activity Constructing definitions.

Note: We also consider a "barely-not-minimal" set of properties as economical definitions, such as the example of the rectangle definition "Quadrilateral with four right angles", since it is enough to verify the existence of three right angles to ensure that it is a rectangle.

4 Results and discussion

4.1 Analysing definitions

Diagnostic task. The diagnostic test presented in the first lesson contained a question about the square definition. This quadrilateral was chosen because it is very familiar to prospective teachers and because the analysis of possible definitions does not involve issues of inclusion, which could skew the findings about the underlying reasoning. So the question 1 (Figure 1) proposed the identification of possible definitions for square, given four options:

1. Check all possible definitions for square:
(A) A four-sided polygon having all sides equal.
(B) A four-sided polygon having all sides equal and four equal angles.
(C) Quadrilateral with four axes of symmetry.
(D) Quadrilateral with equal and perpendicular diagonals.

Figure 1. Question 1 from the diagnostic test, concerning the definitions for square

Options A and D are incorrect because the properties are necessary but not sufficient to define the square. In theory, it is easier to identify the statement A as insufficient because there is no reference to four right angles, which is an easily recognizable property of squares. On the contrary, statement D can lead us directly to squares if we do not remember that, in addition to the properties presented, the diagonals have to intersect at its midpoint. The options B and C are both correct, but we may distinguish them by the different nature of the elements involved. Again, the properties of the statement B are easier to recognize than of those in statement C and, besides that, the definition B is the one commonly used in textbooks.

The results regarding the choice of each option can be found in Table 3 and we will comment on some aspects worth noting. First, analysing the choices one by one, it is clear that all prospective teachers found the definition B to be correct, but only 57% did so for the other correct definition (C). Secondly, the wrong options (A and D) are also selected by many participants, but again with a significant difference from each other. It is worth noting here that, in both cases, the definitions are incorrect for the same reason – the properties presented are necessary but not sufficient.

Option	**%**
A	86
B	100
C	57
D	54

Table 3. Results of Question 1 from the diagnostic test (%).

Thus, a global analysis of the results presented in the Table 3 show that the choices are divided in two groups – the one referring to options A and B, chosen by almost every participants, and the other referring to options C and D, picked by approximately half of the participants. However, these two groups are not separated by its validity (each one contains a wright and a wrong answer), but by the nature of elements they refer to – definitions referring to visible elements (sides and angles) in the case of A and B, and definitions referring to hidden elements (diagonals and axes of symmetry) in C and D options. A lower representation of these options may be due to two reasons: a greater difficulty in recognizing these properties (a difficulty of a conceptual and cognitive nature, associated with the concept of square and the knowledge and skills of prospective teachers) and the awareness that diagonals and axes of symmetry are not commonly used in the definition of square that is used in textbooks (a difficulty of a meta-conceptual nature, associated with the concept that the prospective teachers have about what is a definition in geometry). This difficulty could mean that the participants have trouble in recognizing the arbitrary nature of the definitions, that is, recognizing that any equivalent set of conditions may serve as a definition. Regarding the first two options (on visible elements), its wide choice may show a tendency to accept as definitions, every statement that contains properties that are known or easily recognizable. Furthermore, the joint selection of options A and B still means that 86% of participants appear not identify any conflict in accepting different definitions for the same object, when one clearly corresponds to a restriction of the other.

A more detailed analysis of all the options chosen by the participants reveals the following aspects: a) only 4% of the participants identified exactly the two correct options; b) only 8% chose B as the unique option; and c) 32% chose all options.

Interview. After the questionnaire, at the beginning of the study of quadrilaterals (during the investigation of the properties using Geogebra), we interviewed four participants and ask them to explain their options regarding question 1 from the questionnaire. The interview with Cristina illustrates the kind of reasoning developed by the participants:

Teacher: Now let's see the question 1. You picked A, B and D as definitions for square. I would like that you explain to me why you chose those options.

Cristina: So "A four-sided polygon having all sides equal" [reads], the square has four equal sides; "all sides equal and four [equal] angles" because the square has all the angles measuring 90°; and "equal and perpendicular diagonals"... Because ...

Teacher: Because actually we verify it, is it?

Cristina: Yes ...

Teacher: In fact, you chose the statements that you think that are true for square, is it?

Cristina: Yes ... But not in C.

Teacher: Why?

Cristina: I don't know, I don't remember ... Because I thought there were only two [symmetry axes]. But now I think there are more ... after the lesson [on the investigation of the properties of quadrilaterals].

In this interview, the analysis that Cristina shows is reduced to the verification of the properties listed in every option, which corresponds to examine whether the conditions are necessary. Although she never refers explicitly the requirements of a definition, her justifications for the options she chose reinforce the conjecture that, at the beginning of the unit, the participants considered that the definition of a geometric figure corresponds to a presentation of some properties of the figure, without the awareness that the set of properties has be sufficient to identify the figure or the class of figures. In addition, this dialog also shows that the use of properties involving diagonals and symmetry axes can be an obstacle to the acceptance of the definition, by the increased difficulty in recognizing its veracity.

Definition task. After the interview, the participants worked on a classification task followed by the definition task in which we will focus from now on. As usual, the teacher gave an oral introduction of the task, which was intended to discuss the meaning of a definition. Starting from few contributions of the class, the teacher approached the imperative criteria for a definition to be correct – containing a set of necessary and sufficient conditions – and also the interest of using economic definitions.

Then the group returned to analyse possible definitions for square. The statements presented in the task (Figure 2) turned, on one hand, to a previously identified difficulty – the tendency to accept as definition any statement that present properties that squares actually verify (necessary conditions) – and, on the other hand, focused on the concept of economical definition, which the participants appeared to have only an intuitive idea.

1. Consider the following sentences proposed by a group of pupils as possible definitions for square.
   Definition A: Quadrilateral having all equal sides, parallel 2 by 2 and all equal angles.
   Definition B: Polygon having four sides and two equal diagonals.
   (a) Do you think these definitions are correct?
   (b) Are there any correct and economical definitions?

Figure 2. Question 1 from the definition task.

The activity of the groups was preceded by a moment of individual work on the task, followed by a discussion within each group. The following excerpt relates to a dialogue in the group of Tita, Helena, Cristina and Fernanda:

Tita: So, let's start the discussion. 1a. "Do you think these definitions are correct?" [Read the statement]. [Definition] B, in my opinion, is incorrect.

Group: Yes. Hum, hum.

Tita: Because it fits many [figures]. Not only the square.

Fernanda: Yes. On the other hand, A [definition] refers only to squares.

Helena: A [definition] is correct because "it does not cause ambiguity" [reads what she wrote].

Group: Yes. Very nice!

In this dialogue, the group focus immediately on an aspect previously ignored: the set of properties has to be sufficient. This does not mean that internally they have not identified that they are also necessary, since the properties that are used are known facts that do not require a verification effort, but this aspect is not disclosed in the dialogue. To reach the conclusion that the definition is not correct, the prospective teachers show they have generated corresponding mental images to other geometric figures that meet the definition presented. Thus, we see that the participants show a more correct conception of a definition from the analysis they perform, but also from the comments of meta-conceptual nature. In fact, from the claim that the definition "does not cause ambiguity", we identify the explicit concern to attend to what should be a correct definition. Thus, the dialogue from this group shows an evolution relating to the initial tendency to accept any statement containing necessary properties as a valid definition. This evolution was also observed in the responses of the majority of the prospective teachers, but not all, as shown in the following excerpt from the collective discussion:

Teacher: And now B [statement]. B says "Polygon having four sides and two equal diagonals". If we have a polygon having four sides and two equal diagonals we have . . . Well, a square meets this property or not? A square has four equal sides and diagonals, is not it? But do we have an option other than the square?

Class: Yes, the rectangle.

Teacher: The rectangle, exactly. Therefore, B is not wright.

Ana: Yes, but we wrote that was correct but it was incomplete because it could lead to a confusion in identifying the desired figure.

Teacher: Then, if it can lead to confusion, it is because it is incorrect.

By classifying the definition as "incomplete", Ana (and her group) already shows some knowledge about the need for the conditions to be sufficient in order to correctly identify the figure. Her agreement about the rectangle meeting the definition shows that her analysis also generated figures that correspond to the condition. Therefore, the conclusion of the group seems to be diverging from their colleagues not so much because they analysed it incorrectly, but because they could not completely abandon their previous conception about a definition. Returning to the group of Helena, Tita, Fernanda and Cristina, the prospective teachers discussed the statement A regarding the economic feature.

Helena: Ok. Question 1b.

Tita: I said "in my opinion, the definition A is correct but not economical. It is correct because it only defines the square, presenting features that together are unique to the square." [Reads].

Fernanda: But A is not correct and economic?

Helena: No, I said that although it is correct, I think it is not economic because it has information that is unnecessary.

Tita: Yes, we can say less features. For example "a square is a quadrilateral with all sides equal and all angles equal."

Cristina: You don't need to say "parallel two by two".

Fernanda: Hum ... I understood...

In this dialogue, the group continues to reveal concern in satisfying the requirements of a definition, in particular of an economical definition, so many of their considerations are from a meta-conceptual nature. Participants show they understood the concept of economical definition, although Fernanda needed help from her colleagues. The economical aspect leads them to consider whether the set of conditions is sufficient and minimal, in particular the irrelevance of the parallelism property. However, through their speech is not possible to see how they performed the analysis, that is, we do not know if they deduced the implication (the property "equal sides" and the property "equal angles" both imply that the sides are parallel), or excluded property "parallel sides two by two" and mentally generated the corresponding figure for the new definition.

5 Constructing definitions

Diagnostic task. At this point, we analyse the construction of definitions of quadrilaterals that, unlike the square, mobilizes the concept of inclusion. Since there is a mutual dependence between the process of defining and classifying (de Villiers et al., 2009), it is interesting to characterize the initial knowledge of participants on the classification of quadrilaterals. For that, we turn again to a question of the diagnostic test (Figure 3) applied in the first lesson, which is restricted to very familiar quadrilaterals:

2. Check the true statements:
   a. All rectangles are squares.
   b. All squares are rectangles.
   c. All rectangles are quadrilaterals.

Figure 3. Question 2 from the diagnostic test, concerning the classification of quadrilaterals.

The results show that, in the beginning of the experience, only 25% of the participants considered that all squares are rectangles (but not the opposite), 7% considered that all rectangles are squares and 93% considered that all rectangles are quadrilaterals. This means that 32% of the prospective teachers recognize the possibility of some kind of inclusion relationship between special kinds of quadrilaterals, although some of them are wrong about the direction of that relation. However, when we change the classes of figures using one which is clearly larger than the other (quadrilaterals vs rectangles), the level of success changes completely, revealing that participants recognize much more easily the inclusion relation in this situation.

Definition task. Going back to the definition task (Figure 4), after analysing given definitions for square (question 1), the participants were asked to:

2. Identify all the properties of rectangles;
3. Propose two different definitions for rectangle;
4. Propose two different definitions for parallelogram.

Figure 4. Question 2, 3 and 4 from the definition task.

In respect to question 2, most of the groups identified correctly all the main properties of rectangles (using sides, angles, diagonals and symmetry). Questions 3 and 4 show that they understood that there is no need to present all properties of an object to define it and most produced correct definitions, which is associated to van Hiele level 2 (Battista, 2009). The next response is an example of a correct definition for rectangle, in which one of the properties is valid but unnecessary:

Group A: Rectangles' properties: 4 right angles; 2 by 2 parallel sides; 2 lines of symmetry; bisected diagonals; congruent diagonals. Definition: quadrilateral with 4 right angles and 2 lines of symmetry.

Although less frequent, some definitions were incorrect:

Group A: Parallelogram: quadrilateral without lines of symmetry.

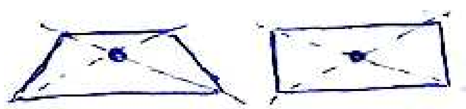
Group B: A parallelogram is a figure composed by 2 paires of congruent and parallel sides, forming 2 acute angles (opposite) and another 2 obtuse (opposite).

Group E: Rectangle: The diagonals intercept in the centre but are not perpendicular; 2 symmetry lines (1 horizontal, 1 vertical) passing in the centre of the figure.

Group F: Rectangle: Geometric figure with 4 sides where the length should be bigger than the height. Parallelogram: Geometric figure similar to rectangle, where the shorter lines are oblique.

The definitions for parallelogram proposed by groups A and B exclude all rhombuses in the first case and all the rectangles in the second, so their definitions are not inclusive. Similarly, the first definition presented by group E excludes squares. These examples show some difficulty to abandon previous conceptions and recognize the hierarchical organization of quadrilaterals. Still in group E, the second definition is incorrect because it does not exclude some rhombuses. Yet, the more striking feature of this definition is that it is dependent of the position that rectangles are usually presented. Group F's response is the only one that considers as properties the relations between the dimension of the sides and their position. Although incorrect, these definitions were presented collectively, which led into an important discussion. Some prospective teachers argued about their validity giving counter-examples or correcting the statements and others noticed and reflected on their own misunderstandings. Finally, some examples of economical definitions demonstrate an interesting analysis, where participants used less usual properties they discovered with Geogebra:

Group C: Rectangle: Quadrilateral with two congruent and bisecting diagonals.



Parallelogram: Each diagonal divides it into congruent triangles.

Group D: Rectangle: Quadrilateral with 2 lines of symmetry passing through the middle points of opposite sides.

In the first definition of group C, the prospective teachers draw a quadrilateral where the diagonals do not bisect so they justify the need to include this property. The second, although roughly written, is very interesting because the word "each" makes a difference (one diagonal would not be enough because of kites). Group D presents a definition focused on the lines of symmetry, but stating their position which is necessary (all rhombuses have also two lines of symmetry in a different location).

Overall, we found four types of problems. Producing economical definitions was the most common difficulty and the hardest to overcome, especially because the participants did not know how to be sure that the properties were sufficient to identify each quadrilateral. A second problem that came up some times was the production of non-inclusive definitions. Even for participants that seemed to understand previously the hierarchical relation between quadrilaterals, sometimes they stopped to consider it, showing difficulties to let go previous conceptions. All these cases correspond to van Hiele level 2, according do Battista (2009). The third problem, happened in very few cases and corresponds to definitions linked to certain positions or relations between parts of the quadrilaterals, clearly associated to frequent prototypes (corresponding to van Hiele level 1). Despite their low frequency, these cases must keep us aware of how striking the systematic exposure to rigid prototypes may be (Yu, Barret, & Presmeg, 2009). Finally, there was only one definition containing an insufficient property to define the quadrilateral.

The previous examples demonstrate some difficulties, but also some interesting successes if we remember that it was the first time that these participants defined something. To formulate definitions implies to investigate invariants. We must identify the common properties to all the elements we include in that class, mobilizing inductive reasoning and visual abilities, in particular visual discrimination and perceptual constancy (Gutierrez, 1996). So, given the fact that most of the definitions were correct, we consider that as a positive indicator regarding those abilities and inductive reasoning. The few participants that produced economical definitions moved to van Hiele level 3 (Battista, 2009) and showed a significant improvement. Given the fact that formulating economical definitions involves also deductive reasoning, it appears the participants showed more difficulty in it.

The construction of definitions was a good opportunity for the participants to learn about the quadrilaterals and to revise their conceptions about the process of defining, as this reflection shows:

> This task raised some doubts because, before we done it, I thought I knew the definitions of each figure, I thought there existed only one for each figure... I came across basic definitions about square or rectangle completely different from what I learned until then. To define figures I never had use angles, diagonals or even lines of symmetry; indeed, I was unaware of their major role. (Reflection written in the participant's portfolio.)

6 Conclusion

In the beginning of the experience, the prospective teachers showed weak knowledge about quadrilaterals, their properties (especially the ones that refer to diagonals and symmetry) and their relations. Their conception about a definition of a figure valued the necessary properties of the figure, but neglected that the set of those properties has to be sufficient and ignored the idea of an economical definition.

However, the work on the sequence of tasks (investigating the properties of quadrilaterals, classifying, and defining) seems to have promoted their reasoning and the reconstruction of their knowledge. Particularly, the definition task gave an opportunity for participants to discuss what a definition is, learn about economical definitions, analyse given definitions, and construct definitions.

Regarding the analyses of definitions, the participants started from an analysis strictly focused on the necessary properties of the figure, by checking if the given properties applied to the figure in question. That verification seems to be simple when the properties relate to visible elements or are well known facts about the figure; on the contrary, the use of properties related to symmetry or diagonals presented an obstacle. This may relate to a problem found by Zaslavsky and Shir (2005) about the arbitrary character of definitions. In the final stage of the sequence of tasks, the participants also took into account that the set of properties has to be sufficient to identify the defined figure. Their discourse included meta-conceptual comments, revealing the development of their knowledge about definitions.

Regarding the construction of definitions, the participants presented mostly correct definitions using properties that they previously ignored, showing the comprehension of the underlying concepts and properties, which supports the importance of the process of defining argued by many researchers (de Villiers et al., 2009; Mariotti & Fischbein, 1997; Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008) However, the participants produced economical definitions in few cases, suggesting that they perform better in inductive rather than deductive reasoning. Classifications associated with the constructed definitions showed, in some cases, a conflict between prior classifications, based on perception, and structural criteria that rules geometrical classifications, which is fundamental to the learning process (also a result indicated by Mariotti and Fischbein, 1997).

Going back to the conjecture that drove our design experiment, the results of this study lead us to conclude that the exploratory work in which the participants engaged allowed them to investigate and discuss their findings and to construct their knowledge in a meaningful way, as well as develop their reasoning. As the testimony of a prospective teacher shows, reflection may play an important role in becoming aware of personal preconceptions and knowledge, which is an essential part of teacher education (Ponte & Chapman, 2008). Regarding the activity of defining, initially the participants showed weaker knowledge than we expected. The understanding of the concept of definition and, especially of an economical definition, seems to be difficult for many participants but, nevertheless, a fruitful one. On one hand, the analysis and discussion of given definitions played a very important role in learning about this process, by allowing participants to realize how a true statement might be an incorrect definition. On the other hand, the construction of definitions also mobilized reasoning and the demand for different hypotheses encouraged the use of properties requiring a variety of elements (such as diagonals and symmetry axes), favouring the comprehension of the arbitrary character of definitions and a deeper understanding of the quadrilaterals. Finally, the implementation of the definition task after investigating and classifying quadrilaterals encouraged the prospective teachers to reason according to the established classifications, which was very challenging.

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Tworzenie definicji czworokątów przez przyszłych nauczycieli – podejście poprzez odkrywanie

Streszczenie

Artykuł dotyczy kształcenia geometrycznego przyszłych nauczycieli szkoły podstawowej, odbywającego się w Portugalii w ramach kursu geometrii, prowadzonego przez pierwszego autora niniejszego opracowania. Naszym celem jest scharakteryzowanie zajęć dotyczących definiowania czworokątów, z wykorzystaniem specjalnie skonstruowanych zadań rozwiązywanych podczas kursu geometrii. Sam kurs był realizowany podczas drugiego roku studiów nauczycielskich a jego konstrukcja była specjalnie zaprojektowana. Zakładamy, że nauczanie-uczenie się opiera się na odkrywaniu, z wykorzystaniem refleksji przyszłych nauczycieli nad procesem własnego uczenia się. Dodatkowo uważamy, że zajęcia polegające na definiowaniu czworokątów odgrywają dużą rolę w rozwijaniu rozumowań, pogłębiają wiedzę o ich własnościach i wspierają pełniejsze spojrzenie na matematykę.

Dane były czerpane ze sprawozdań umieszczanych w protfolio uczestników kursu, z nagrań audio i wideo tworzonych podczas dyskusji odbywających się w trakcie kursu. Wyniki pokazują, że początkowo uczestnicy kursu mieli fałszywe rozumienie czym jest definicja, skupiali się jedynie na własnościach figur.

Jednakże pod koniec pracy nad sekwencją zadań byli w stanie wziąć pod uwagę fakt, że zbiór własności powinien być wystarczający dla zidentyfikowania definiowanej figury, a to wskazuje na rozwój ich wiedzy. Pod koniec realizowania sekwencji większość uczestników była w stanie zaprezentować poprawną definicję, z wykorzystaniem tych własności które wcześniej były przez nich ignorowane, okazując zrozumienie samych pojęć . W kilku przypadkach stworzyli ekonomiczną definicję (czyli posiadającą tylko konieczną i wystarczającą liczbę warunków) i lepiej realizowali rozumowanie indukcyjne niż dedukcyjne.

Wnioskujemy więc, że praca polegająca na odkrywaniu w znacznym stopniu umożliwiła uczestnikom konstrukcję własnej wiedzy, zaś refleksja odegrała istotną rolę w uświadomieniu sobie własnych wcześniejszych sądów i wcześniejszej wiedzy. Na początku uczestnicy analizowali definicje skupiając się dokładnie na koniecznych własnościach figury, które wydają się proste gdy dotyczą widzialnych elementów lub są związane z dobrze znanymi faktami. W końcowym etapie rozwiązywania sekwencji zadań uczestnicy brali również pod uwagę, że zbiór własności musi być wystarczający dla skonstruowania figury, zaś podczas tworzenia definicji prezentowali na ogół poprawne definicje z wykorzystaniem tych własności które wcześniej ignorowali, wskazując przez to na zrozumienie danego pojęcia i jego własności.

Przeprowadzając taki proces w etapie, kiedy studenci mają już za sobą badanie i klasyfikację czworokątów, wspieramy przyszłych nauczycieli w korzystaniu z wielu własności i pokazujemy podstawy ustalonych klasyfikacji, co było bardzo trudnym zadaniem.

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